

Gauge independence of tunneling rates

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Abstract

Despite the gauge dependence of the effective action at zero and finite temperature, it is shown that tunneling and nucleation rates remain independent of the choice of gauge-fixing. Taking as a starting point the path integral that defines the transition amplitude from a false vacuum onto itself, it is shown that decay rates are exactly determined by a non-convex, false-vacuum effective action evaluated at an extremum. The latter can be viewed as a generalized bounce configuration, and gauge-independence follows from the appropriate Nielsen identities. This holds for any election of gauge-fixing that leads to an invertible Faddeev-Popov matrix. The result, which is nonperturbative and model-independent, clarifies issues of convexity, as well as how to incorporate radiative corrections in tunneling calculations.

1 Introduction

Since the work of Jackiw and Dolan [1, 2], it has been known that the quantum effective action in gauge theories, and in particular its zero momentum piece, the effective potential, depend on the choice of gauge-fixing. The effective potential is used to calculate physically meaningful quantities, both at zero temperature (such as vacuum energies, masses, tunneling rates), as well as at finite temperature (e.g. critical temperatures in phase transitions and their nucleation rates). Given that physical observables cannot depend on the choice of gauge, it becomes important to understand how to extract gauge-independent information from the effective action.

The works of Nielsen, Kugo and Fukuda [3, 4] set the basis for the resolution of these issues, providing identities that encode the behavior of the effective action under changes of the gauge-fixing parameter. Originally derived for specific classes of gauge-fixing functions, these identities have been extended to arbitrary choices of the latter [5, 6]¹. The Nielsen identities imply that the gauge dependence of the effective action is equivalent to a nonlocal field redefinition. For the effective potential, in the case of a single scalar field σ , they adopt the form

$$\left(\xi \frac{\partial}{\partial \xi} + C(\sigma; \xi) \frac{\partial}{\partial \sigma} \right) V_{\text{eff}}(\sigma; \xi) = 0, \quad (1.1)$$

where ξ is the gauge-fixing parameter, V_{eff} the quantum effective potential, and $C(\sigma; \xi)$ is a functional which can be calculated in terms of Feynman diagrams. An immediate consequence of this is that physical quantities defined at extrema of the effective potential, where $\partial V / \partial \sigma = 0$, become gauge-independent [3, 9]. This is the case for vacuum energies, as well as masses of scalar fluctuations around vacua. However, vacuum expectation values of fields, as well as the values of the effective potential in between minima, remain gauge-dependent and hence unphysical. The Nielsen identities have been explicitly studied and verified for several theories, both at zero and finite temperature, mostly for scalar QED in a variety of gauges (see e.g. [9–12, 14–20]), but also in the Standard Model [21]. Being a nonperturbative result, some care has to be taken to define a perturbative counting scheme such that the identities hold order-by-order. This can happen for example with the vacuum energy in a truncated perturbative calculation. Given that the minimization conditions may enforce a relation between powers of the tree-level couplings of the theory that differs from the usual loop counting (such as in Coleman-Weinberg models of radiative symmetry breaking [22]), some resummation of loop effects might be needed to explicitly check the gauge independence of the vacuum energy to a given level of approximation [20, 21, 23].

The issue of the gauge dependence of vacuum energies and physical masses being solved by the Nielsen identities, one may worry that the gauge parameter might still work its way into calculations that depend on the values of the potential away from the vacua, such as tunneling or nucleation rates. These are needed to tackle important questions in particle physics, such as the stability of the Higgs vacuum during and after inflation (see e.g. [24–27] for references including discussions on gauge dependence), or the properties of phase transitions in the early Universe, which can have an impact on baryogenesis (see [28] for a review). Some important studies of the gauge dependence of tunneling amplitudes have been done in previous works, using semiclassical techniques and focusing either in the action of the corresponding Euclidean solution [16, 17], or in the determinants of the fluctuations around the latter [29]. In these works, gauge independence was shown to hold at the lowest nontrivial orders in perturbation theory for specific theories and choices of gauge-fixing, yet these analyses could

¹See also [7–14] for discussions about the validity of the Nielsen identities in a variety of gauges.

not discard the appearance of gauge dependence at higher orders. Reference [19] found a nonzero gauge dependence of tunneling rates in the Abelian Higgs model, but this was interpreted as a possible effect of the breaking of the derivative expansion of the effective action. As a possible solution to the issue of the gauge dependence of the effective action, which would seem to allow to compute gauge-independent tunneling rates, Nielsen has advocated [30] for the use of a potential obtained by performing a field redefinition that compensates the gauge dependence of (1.1). A related simplified approach, in which the scalar fields in the effective action are canonically normalized by absorbing field-renormalization factors, was used in [26,27,31]. Still, it remains unclear to see how this redefined potential could arise in the calculation of tunneling rates from first principles.

The problem of the gauge dependence of tunneling rates is connected to that of including quantum corrections in a consistent manner in the calculation of tunneling amplitudes. In the usual computations by means of a saddle point expansion of the path integral, the role of the effective potential—which itself includes quantum corrections—is in principle unclear. This becomes especially problematic in theories with vacua that only arise radiatively, with the classical potential appearing inside the path integral having no nontrivial extrema. From early on it was assumed that the right answer involved using the effective potential, rather than its classical counterpart [32], though it was not until the work of [33] that the correctness of this procedure was justified in part. There it was argued that one could compute tunneling rates by doing the usual semiclassical expansion in an effective theory obtained by integrating out the gauge fields. However, it was also noted that the resulting effective potential does not exactly match the full effective potential of the theory, among other things because it is obtained from connected, rather than one-particle-irreducible Green functions. Thus the exact role played by the effective potential, as well as the consequences of its gauge dependence, remained unclear. Very recently, a formalism for consistently calculating tunneling rates by performing a saddle point evaluation of the path integral around the quantum, rather than classical path, was developed and applied in [34–36]. Again, a clear justification for the use of the quantum path, and an understanding of the ensuing consequences for the gauge dependence of the results when considering theories with gauge fields, is still missing.

A further puzzle related with the possible role of the effective action in the computation of tunneling rates is related to its known reality and convexity properties [37]. The convexity of the effective action implies concavity of the effective potential, which thus cannot have false vacua. Furthermore, the true effective potential lacks an imaginary part, which would be associated with an unstable state. This suggests that another quantum functional must play a role in the calculation of tunneling rates, such as one of the “localized” effective actions proposed by Weinberg and Wu [38].

To the best of our knowledge, a nonperturbative result concerning the gauge independence of tunneling rates and their relation to the effective action is lacking, and in this paper we remedy this by providing a simple derivation based on Callan and Coleman’s path integral definition of the tunneling rate [39], carefully treating the boundary conditions and gauge-fixing, and using the Nielsen identities. It turns out that the exact tunneling rate can be simply expressed in terms of the imaginary part of a non-convex Euclidean effective action, constructed from the transition amplitude from the false vacuum onto itself, (rather than the corresponding amplitude for the true vacuum), and evaluated at a generalized bounce solution to its quantum equations of motion. This validates the approach of [34–36], which is shown to enforce the correct boundary conditions in the vacuum-to-vacuum path integral. Gauge-independence follows from the Nielsen identities of the false vacuum effective action, which imply that its value on extremal configurations does not depend on the gauge parameters.

Much like S-matrix elements are independent of the choice of gauge, and may be calculated with an arbitrary choice of gauge-fixing, tunneling rates can be computed in any gauge. Nielsen's field redefinition for arbitrary gauges can be thought of as a transformation of the fields which takes them to a reference gauge slice, and is not essential for achieving gauge-independent results. The cancellation of the gauge dependence is automatic –up to higher order effects in a perturbative truncation– as long as the effective action is evaluated consistently, including derivative terms.

2 Gauge independence of tunneling rates

We will provide a derivation of the gauge independence of tunneling rates by studying the properties of vacuum functionals defined in terms of path integrals. These functionals are the vacuum-to-vacuum transition amplitudes in the presence of sources –both for the true vacuum and for a false vacuum– and their associated effective actions, obtained by means of Legendre transformations. We will emphasize that decay rates are associated with the false vacuum functionals, rather than the ones corresponding to the true vacuum of the theory. This explains why one can consistently consider false vacua and their decay rates, including radiative corrections, despite the reality and convexity of the true vacuum effective action, which prevent it from playing a role in the calculation of decay rates. Next we will review the derivation of the Nielsen identities for the effective actions, following reference [6], and then move on to consider vacuum decay rates, relating them to the false vacuum effective action. The gauge independence of tunneling rates will follow as a consequence of the latter functional's Nielsen identities.

In the spirit of reference [6], we may consider a theory with fields labelled in DeWitt's compact notation [40] as $\phi \equiv \{\phi_j\}$, with the index j referring to any continuous or discrete degree of freedom, including space-time dependence. In the presence of a gauge symmetry with a Lie Algebra \mathfrak{g} spanned by generators $T^a, a = 1 \cdots \dim(\mathfrak{g})$, there will be gauge transformations under which the classical action will be invariant. These transformations depend on a gauge parameter $\alpha = \alpha^a T^a$, and can be written as²

$$\delta\phi_j \equiv D_j^a[\phi]\alpha^a. \quad (2.1)$$

We will start by considering the vacuum-to-vacuum transition amplitude in the presence of a source, $Z[J]$. We will assume that the source produces the same perturbation, yielding to the same ground-state, at times $t = \pm\infty$. The functional $Z[J]$ is related by the generator $W[J]$ of connected amplitudes as $Z[J] = \exp iW[J]$. Introducing a complete basis of Heisenberg-picture, time-independent eigenstates $|q\rangle$ of the field operators $\hat{\phi}$, such that $\hat{\phi}_i|q\rangle = q_i|q\rangle$, the identity operator can be written as

$$I = \int [dq] \mu(q) |q\rangle \langle q|, \quad (2.2)$$

where μ is an integration measure. Using the above spectral decomposition, we may write the vacuum-to-vacuum amplitude as

$$\begin{aligned} Z[J] &= \exp iW[J] = \lim_{T \rightarrow \infty} \langle 0 | e^{-iHT} | 0 \rangle^J = \lim_{T \rightarrow \infty} \int [dq][dq'] \mu(q) \mu(q') \langle 0 | q \rangle^J \langle q | e^{-iHT} | q' \rangle^J \langle q' | 0 \rangle^J \\ &= \int [dq][dq'] \mu(q) \mu(q') \psi_0^J(q') \psi_0^{J*}(q) \int_{q'}^q [d\phi] \mu(\phi) \exp i \left[\tilde{S}_g[\phi; \xi] + J_j \phi^j \right] \equiv \langle \exp[iJ_j \phi^j] \rangle. \end{aligned} \quad (2.3)$$

²To ease the notation we assume a simple gauge group with a positive definite metric acting on the Lie Algebra, and work in the basis in which it is given by the identity. Hence we do not need to distinguish between upper and lower indices in the Lie Algebra, though we keep the distinction in the field indices in DeWitt's notation.

In the above equation, we remind the reader that the trace over j indices includes an integration over space-time coordinates. $\psi_0^J(q) = \langle q|0\rangle^J$ can be understood as a field-space wave-function of the vacuum state in the Heisenberg picture and in the presence of the source J . The integration measure μ is required to satisfy [6]

$$\mu_{,j} D^{aj} + \mu D_{,j}^{aj} = 0. \quad (2.4)$$

This happens for example in dimensional regularization (DR) with a constant μ , since the D^{aj} are linear in the fields, and then $D_{,j}^{aj}$ becomes an integral of a constant function which vanishes in DR. $\tilde{S}_g[\phi; \xi]$ in equation (2.3) is given by the classical action plus a gauge-fixing piece, depending on a gauge-parameter ξ , on which we will elaborate later. Finally, the notation for the integration symbol in ϕ in (2.3) alludes to the fact that the fields must satisfy the following boundary conditions,

$$\lim_{t \rightarrow -\infty} \phi = q', \quad \lim_{t \rightarrow \infty} \phi = q. \quad (2.5)$$

Since the vacuum-to-vacuum transition amplitude in the absence of a source is one, up to a phase,³ the wave-function and path integrations define a measure which allows to identify $Z[J]$ with an average, as indicated at the end of equation (2.3). From this, using Hölder's inequality, reference [37] argued that the generator of connected diagrams $W[J] = -i \log Z[J]$ is a real, concave functional, i.e. satisfying $(1 - \alpha)W[J_1] + \alpha W[J_2] \geq W[(1 - \alpha)J_1 + \alpha J_2]$ for $0 \leq \alpha \leq 1$.

Typically, it is assumed that the wave-function of the vacuum peaks around a single point in field-space, $\psi_0^J(q) \sim \delta(q - q_0^J)$, so that $Z[J]$ can be expressed as a single path integral

$$Z[J] \approx \int_{q_0^J}^{q_0^J} [d\phi] \mu(\phi) \exp i \left[\tilde{S}_g[\phi; \xi] + J_j \phi^j \right]. \quad (2.6)$$

However, this approximation will fail in the presence of N multiple vacua, in which case one expects the true vacuum's wave-function to peak around the field configurations $q_0^{J,m}$, $m = 1, \dots, N$ of the local vacua, in which case $Z[J]$ will be better approximated by a sum of path integrals, as in

$$Z[J] \approx \sum_{m,n=1}^N Z^{m,n}[J], \quad Z^{m,n}[J] = \mathcal{N}_{mn} \int_{q_0^{J,m}}^{q_0^{J,n}} [d\phi] \mu(\phi) \exp i \left[\tilde{S}_g[\phi; \xi] + J_j \phi^j \right], \quad (2.7)$$

where the \mathcal{N}_{mn} are normalization constants related with the size of the vacuum wave-function's peaks on top of the different vacua. It is expected that a single $Z^{m,n}[J]$ will fail to yield a concave $W[J]$, and yet the sum of the $Z^{m,n}[J]$ will do whenever it is a good approximation to the full $Z[J]$.

Aside from $Z[J]$, one may introduce an analogous functional corresponding to the transition of an unstable state (or false vacuum) onto itself. Denoting this state by $|F\rangle$ and its field-space wave-function by ψ_F , then we may write, in analogy with equation (2.3),

$$Z_F[J] = \lim_{T \rightarrow \infty} \langle F | e^{-iHT} | F \rangle^J = \int [dq][dq'] \mu(q) \mu(q') \psi_F^J(q') \psi_F^{J*}(q) \int_{q'}^q [d\phi] \mu(\phi) \exp i \left[\tilde{S}_g[\phi; \xi] + J_j \phi^j \right]. \quad (2.8)$$

Crucially, since the state is unstable and decays, its transition amplitude back to itself is not the identity up to a phase, and thus one cannot conclude that $W_F[J]$ is a concave functional. Neither it is guaranteed to be real, since the Hamiltonian acting on the unstable $|F\rangle$ will pick an imaginary

³At least in the case of a time-independent source.

part. In fact, it is this functional $Z_F[J]$ which allows to calculate the decay rate of the false vacuum. The reason is that, as follows from equation (2.8), $Z_F[0]$ can be written as

$$Z_F[0] = \lim_{T \rightarrow \infty} \langle F | e^{-iHT} | F \rangle \sim e^{-i\epsilon VT}, \quad (2.9)$$

where ϵ denotes the false vacuum energy density. An instability is signalled by an imaginary part of ϵ , which yields an associated decay rate

$$\gamma = -2 \text{Im} \epsilon = -\frac{2}{VT} \text{Re} (\log Z_F[0]). \quad (2.10)$$

Note that an unstable vacuum is associated with an imaginary $W_F[0] = -i \log Z_F[0]$, in contrast to the true-vacuum functional $W[0]$ which remains real. Again, whenever the false vacuum's wave-function peaks at a field configuration q_F , one may approximate $Z_F[J]$ by a single path integral,

$$Z_F[J] \approx \int_{q_F^J}^{q_F^J} [d\phi] \mu(\phi) \exp i \left[\tilde{S}_g[\phi; \xi] + J\phi \right]. \quad (2.11)$$

As was commented in regards to equation (2.7), a single path integral will fail to yield a concave functional $W_F[J]$, as is expected for the false-vacuum.

From the above vacuum functionals, one may construct effective action functionals that depend on the mean fields by performing Legendre transformations. The usual mean field $\bar{\phi}_j \equiv \langle \phi_j \rangle^J$ represents the expectation value of the field ϕ_j in the groundstate and in the presence of a source, and is defined as

$$\bar{\phi}_i = \langle \phi_i \rangle^J = \frac{\delta W[J]}{\delta J^i} = e^{-iW} \sum_{m,n=1}^N \mathcal{N}_{mn} \int_{q_0^{J,m}}^{q_0^{J,n}} [d\phi] \mu(\phi) \phi_i \exp i \left(\tilde{S}_g + J^j \phi_j \right), \quad (2.12)$$

where we have used the approximation of equation (2.7), in which the vacuum functional is given by a sum of path integrals with boundary conditions determined by the N peaks of the vacuum wave-function. One may also define a false vacuum mean-field $\bar{\phi}_F$, which, using the approximation (2.11), will be given by

$$\bar{\phi}_{Fi} = \langle \phi_i \rangle_F^J = \frac{\delta W_F[J]}{\delta J^i} = e^{-iW_F} \int_{q_F^J}^{q_F^J} [d\phi] \mu(\phi) \phi_i \exp i \left(\tilde{S}_g + J^j \phi_j \right). \quad (2.13)$$

The effective action Γ is given by

$$\Gamma[\bar{\phi}] = W[J] - J_j \bar{\phi}^j, \quad (2.14)$$

where it is understood that the mean fields and the sources are related by the following identities,

$$\bar{\phi}_j = \frac{\delta W[J]}{\delta J^j}, \quad J_j = -\frac{\delta \Gamma[\phi]}{\delta \phi^j} \equiv -\Gamma_{,j}. \quad (2.15)$$

If the vacuum functional can be written as in equation (2.7), it can be seen that Γ may be implicitly defined by the following sum of path integrals (where we generalize the single path-integral results of references [41–43] to account for a multi-peaked vacuum wave-function):

$$\exp i\Gamma[\bar{\phi}; \xi] = \sum_{m,n=1}^N \mathcal{N}_{mn} \int_{q_0^{J,m}-\bar{\phi}_\infty}^{q_0^{J,n}-\bar{\phi}_\infty} [d\phi] \mu(\phi) \exp i \left[S_g[\bar{\phi}, \phi; \xi] - \Gamma_{,j}[\bar{\phi}; \xi] \phi^j \right]. \quad (2.16)$$

Note how the boundary conditions in the sum of path integrals depend on the combinations $q_0^{J,m} - \bar{\phi}_\infty$, where $\bar{\phi}_\infty$ represent the limiting values of $\bar{\phi}$ at $t \rightarrow \pm\infty$ (we are considering sources which lead to the same groundstate at those times). This arises after performing field-redefinitions inside the path integrals that define $Z[J]$, which is reflected by the change in notation in S_g , such that $S_g[\phi, \bar{\phi}; \xi] = \tilde{S}_g[\bar{\phi} + \phi; \xi]$, with \tilde{S}_g appearing in equations (2.3), (2.7), (2.8), (2.11), (2.12), (2.13). Given the relation between J and $\bar{\phi}$ enforced by equation (2.15), the $q_0^{J,m}$ can be expressed as a function of the mean field $\bar{\phi}$. Using the definition (2.14) of the effective action, it can also be seen that (2.12) can be rewritten in terms of Γ as

$$\bar{\phi}_i = \langle \phi_i \rangle^J = e^{-i\Gamma} \sum_{m,n=1}^N \mathcal{N}_{mn} \int_{q_0^{J,m}}^{q_0^{J,n}} [d\phi] \mu(\phi) \phi_i \exp i \left[\tilde{S}_g[\phi; \xi] - \Gamma_{,j}(\phi^j - \bar{\phi}^j) \right]. \quad (2.17)$$

Given the concavity of $W[J]$, $\delta W/\delta J$ has a monotonous dependence on J and is thus a single-valued functional. This implies that $\tilde{\Gamma}[J, \bar{\varphi}] \equiv W[J] - J\bar{\varphi}$, when considered as a function of J for a fixed $\bar{\varphi}$ (with J and $\bar{\varphi}$ unrelated), is concave and has a unique minimum at J satisfying $\delta W/\delta J = \bar{\varphi}$, so that

$$\Gamma[\bar{\phi}] = \min_J \tilde{\Gamma}[J, \bar{\phi}]. \quad (2.18)$$

From this one can infer that $\Gamma[\bar{\phi}]$ is itself a convex functional of $\bar{\phi}$ [37]. For a constant field $\bar{\phi}$, the effective potential is defined as

$$\Gamma[\bar{\phi}] = - \int d^4x V_{\text{eff}}[\phi], \quad (2.19)$$

which implies that V_{eff} is a concave functional. As noted by Weinberg and Wu in reference [38], this is not in contradiction with the existence of false, unstable vacua, which in principle require a potential with alternating positive and negative curvature. The reason is that the effective potential evaluated at a field-value $\bar{\phi}$ captures the minimum amount of work needed to change the *ground-state* of the system in the presence of a current enforcing a ground-state expectation value $\langle \phi \rangle = \bar{\phi}(J)$ [44]. Equivalently, the effective potential can be understood in terms of the *minimum* energy density of states $|s\rangle$ with $\langle s|\phi|s\rangle = \bar{\phi}$ [38]. This does not capture the energy density of false, unstable vacua, but rather that of lower-energy superpositions of the multiple local vacuum states. This can be nicely understood from the appearance of multiple path integrals in equation (2.16), showing that the true effective action implies summing over sectors in which the wave-function of the true vacuum has a peak. Traditional perturbative calculations of the effective action involve a single path-integral, and hence represent only a partial contribution to the effective action, and thus fail to be convex. As also noted by Weinberg and Wu, one may identify the usual calculations of the effective potential with a minimization of the energy density of states further constrained to have a small dispersion. In the present discussion this can be immediately understood from the fact that a single path integral with boundary conditions $q_0^{J,m}$ corresponds to wave-functions peaking at $q_0^{J,m}$, which gives a simple functional-integral interpretation of Weinberg and Wu's "localized" effective potential. In fact, it has been shown in explicit calculations in a variety of works that summing over different path integrals (or equivalently expanding around different saddle-points) it is possible to obtain a concave effective potential [45–51]; here we have argued that this sum can be understood as a consequence of a multi-peaked vacuum wave-function. The usual constructions of convex potentials involve only summing over the diagonal $m = n$ contributions in equation (2.16), while our line of reasoning calls for including additional sectors with mixed boundary conditions, corresponding to tunneling effects between the

local vacua. These tunneling effects, however, can be nonperturbatively suppressed with respect to the perturbative contributions of the $m = n$ sectors; the situation would be analogous to the inclusion of instanton corrections in gauge theories, corresponding to tunneling in between topological vacua.

As should be clear from the previous discussions, the real, convex functional Γ cannot play a role in the computation of tunneling rates. However, one may construct an alternate effective action from the false vacuum transition amplitude $Z_F[J]$, and this new functional, being neither convex nor real, will turn out to play a crucial role for tunneling rates. In analogous manner to the definition of Γ , one can define the false vacuum effective action Γ_F as the Legendre transformation of the false vacuum functional $W_F[J]$. Note that such a definition assumes a unique relation between a source J and a false-vacuum mean field $\bar{\phi}_F$. It has been argued that this can be problematic for a non-concave $W[J]$, since $\bar{\phi}(J)$ and $W[J]$ may be multivalued at the classical level [45], with the multivaluedness arising from the existence of different classical vacua in the presence of a source. The definition of $\bar{\phi}_F$ in terms of an expectation value in the false-vacuum, as in equation (2.13), ensures that there is no such ambiguity at the quantum level. Indeed, denoting the false-vacuum state in the Schrödinger picture as $|F, t\rangle$, $\bar{\phi}_F$ is defined in terms of J as

$$\bar{\phi}_F(J) = \frac{\langle F, \infty | \phi | F, -\infty \rangle^J}{\langle F, \infty | F, -\infty \rangle^J}. \quad (2.20)$$

At the level of the path integral definition in (2.13), $\bar{\phi}_F(J)$ only picks up the false-vacuum classical branch, as is clear from the boundary conditions in the integral. In the case of the true vacuum mean field, it is also well defined at the quantum level, as again $\bar{\phi}$ is unambiguously defined as an expectation value in the true-vacuum. However, in this case one may have to sum over different path integrals which capture the different classical branches of the relation between the mean field and the current, as in equation (2.12).

In the single integral approximation of equation (2.11), Γ_F is implicitly defined by

$$\exp i\Gamma_F[\bar{\phi}_F; \xi] = \int_0^1 [d\phi] \mu(\phi) \exp i [S_g[\bar{\phi}_F, \phi; \xi] - \Gamma_{F,j}[\bar{\phi}_F; \xi] \phi^j]. \quad (2.21)$$

The zero boundary conditions can be explained as follows. We are assuming that the source enforces the same false-vacuum state at $t = \pm\infty$, so that the mean field will approach the same value $\bar{\phi}_\infty$ at these times. In a single-path integral approximation, the false-vacuum wave-function in the presence of a source is then expected to peak at this mean value, i.e. $q_F^J = \bar{\phi}_\infty$. The boundary conditions of the path integral in equation (2.21) are the analogues of those in (2.16), but with the $q_0^{J,m}$ reduced to a single q_F^J , giving $q_F^J - \bar{\phi}_\infty = 0$.

Since $W_F[J]$ is not a concave functional, it follows that the resulting effective potential $V_{F\text{eff}}$ (defined similarly to (2.19)) will not be concave, and is expected to have a local minimum related with the false vacuum. In fact Γ_F can be understood as one of Weinberg and Wu's local functionals, corresponding to choosing a “wrong” minimization branch in equation (2.18). The fact that we are choosing the branch that precisely corresponds to the false vacuum is ensured by the definition of Γ_F in terms of Z_F .

Once we have defined the effective action functionals Γ and Γ_F , we may further specify the terms appearing inside the path integrals, and study the gauge dependence. In equations (2.16), (2.21), $S_g[\bar{\phi}, \phi; \xi]$ corresponds to the gauge-fixed action, given by the sum of the usual classical action evaluated on $\bar{\phi} + \phi$, plus a gauge-fixing term $S_{gf}[\bar{\phi}, \phi; \xi]$, and a ghost term $S_{gh}[\bar{\phi}, \phi, \bar{c}, c; \xi]$ depending

on additional Grassmannian ghost fields \bar{c}, c . These contributions adopt the form

$$\begin{aligned} S_g[\bar{\phi}, \phi; \xi] &= S[\bar{\phi} + \phi] + S_{gf}[\bar{\phi}, \phi; \xi] + S_{gh}[\bar{\phi}, \phi; \xi], \\ S_{gf} &= \int d^D x \frac{1}{2\xi} \mathcal{F}^a \mathcal{F}^a, \\ S_{gh} &= \int d^D x \bar{c}^a \mathcal{H}^{ab} c^b, \end{aligned} \quad (2.22)$$

where, using the notation of (2.1) and omitting the dependence on the fields,

$$\mathcal{H}^{ab} = F_{,k}^a D^{bk}. \quad (2.23)$$

In the identities (2.22) and (2.23), \mathcal{F}^a is the gauge-fixing function, which for example has the form $\mathcal{F}^a = \partial_\mu A_\mu^a$ in Fermi gauges (A_μ^a being the gauge field) though more generally it may depend on scalar fields and their expectation values, as in R_ξ gauges.

The Nielsen identities for the effective actions can be derived by studying how the contributions inside the path integrals are modified under changes of ξ . Assuming for simplicity that \mathcal{F}^a is independent of the gauge-fixing parameter (the result for the general case will be given later) and considering an infinitesimal change of ξ , the only contribution inside the path integral that is modified is S_{gf} ,

$$\delta_\xi S_{gf} = -\frac{d\xi}{2\xi^2} \int d^D x \mathcal{F}^a \mathcal{F}^a, \quad (2.24)$$

while under an infinitesimal gauge transformation with parameter $d\alpha$ it behaves as

$$\delta_\alpha S_{gf} = \frac{1}{\xi} \int d^D x \mathcal{F}^a \mathcal{F}_{,j}^a D^{bj} d\alpha^b = \frac{1}{\xi} \int d^D x \mathcal{F}^a \mathcal{H}^{ab} d\alpha^b. \quad (2.25)$$

As noted in [6], the effect of the transformation in (2.24) can be compensated by appropriately engineering a gauge transformation as in equation (2.25). This happens for a choice of gauge parameter

$$d\alpha = \frac{d\xi}{2\xi} \mathcal{H}^{-1} \mathcal{F} \equiv \frac{d\xi}{2\xi} \mathcal{G} \mathcal{F}, \quad (2.26)$$

where we defined $\mathcal{G} = \mathcal{H}^{-1}$. This gauge transformation does not leave the path-integral measure invariant; however, as seen in [6], the corresponding Jacobian exactly cancels the variation of the ghost action S_{gh} , as long as the measure satisfies (2.4). This may be most easily checked by writing the exponential of the ghost integral as a determinant,

$$\int [d\bar{c}][dc] \mu(\bar{c}, c) \exp iS_{gh} = \det \mathcal{H} = \exp \text{tr} \log \mathcal{H}, \quad (2.27)$$

where the trace affects the discrete and spacetime indices of the operator \mathcal{H} of equation (2.23). Under the gauge transformation with parameter (2.26), the classical action $S[\phi]$ remains invariant. Therefore, after combining a variation of ξ with a field redefinition given by the gauge transformation of (2.26), the net effect in equations (2.16), (2.21) is simply a change in the source term, e.g. $-\Gamma_{,j}[\bar{\phi}; \xi] \phi^j$. Focusing on Γ , we may add a further transformation $\phi \rightarrow \phi - \bar{\phi}$, so as to be able to identify operator averages as in (2.17). A subtlety is that the boundary conditions in the path integrals could themselves depend on the gauge parameter, and they are affected by the gauge transformation (2.26). However, it can be easily seen that the path integrals in the $m = n$ sectors are stationary with

respect to infinitesimal variations of the boundary conditions, given that the latter are identical for $T = \pm\infty$. The variations of the $m \neq n$ sectors cancel in pairs if $\mathcal{N}_{mn} = \mathcal{N}_{nm}$. The \mathcal{N}_{mn} are related to areas of the vacuum wave-function under its peaks, and thus should be gauge-independent given their probabilistic interpretation. Going beyond the discrete sum approximation of equation (2.7), the variations in the boundary conditions in the ϕ path integrals under the gauge transformation of equation (2.26) can be absorbed by redefining the q, q' variables, which should not affect the value of the vacuum wave-function, which should be gauge-invariant. With the previous discussion in mind, we can just ignore the effect of infinitesimal changes in the boundary conditions and write, in the discrete sum approximation (although a similar result will hold for the full effective action):

$$\exp i\Gamma[\bar{\phi}; \xi + d\xi] = \sum_{m,n} \mathcal{N}_{mn} \int_{q_0^{J,m}}^{q_0^{J,n}} [d\phi] \mu(\phi) \exp i \left[\tilde{S}_g[\phi; \xi] - \Gamma_{,j}[\bar{\phi}; \xi + d\xi](\phi^j - \bar{\phi}^j) - \frac{d\xi}{2\xi} \Gamma_{,j}[\bar{\phi}; \xi] \tilde{D}^{aj} \tilde{\mathcal{G}}^{ab} \tilde{\mathcal{F}}^b \right], \quad (2.28)$$

where we have ignored contributions of higher order in $d\xi$ where appropriate, and \tilde{D}_j^a , $\tilde{\mathcal{G}}^{ab}$ and $\tilde{\mathcal{F}}^b$ are obtained from D_j^a , \mathcal{G}^{ab} and \mathcal{F}^b after substituting $\phi \rightarrow \bar{\phi} - \phi$. For an infinitesimal $d\xi$ this implies

$$\begin{aligned} \frac{\partial}{\partial \xi} \Gamma &= -e^{-i\Gamma} \sum_{m,n} \mathcal{N}_{mn} \int_{q_0^{J,m}}^{q_0^{J,n}} [d\phi] \mu(\phi) \left(\frac{\partial}{\partial \xi} \Gamma_{,j}(\phi^j - \bar{\phi}^j) + \frac{1}{2\xi} \Gamma_{,j} \tilde{D}^{aj} \tilde{\mathcal{G}}^{ab} \tilde{\mathcal{F}}^b \right) \exp i \left[\tilde{S}_g - \Gamma_{,j}(\phi^j - \bar{\phi}^j) \right] \\ &= - \left\langle \frac{\partial}{\partial \xi} \Gamma_{,j}(\phi^j - \bar{\phi}^j) + \frac{1}{2\xi} \Gamma_{,j} \tilde{D}^{aj} \tilde{\mathcal{G}}^{ab} \tilde{\mathcal{F}}^b \right\rangle, \end{aligned} \quad (2.29)$$

where we used the definition of average of equation (2.17). Using that for the mean field one has $\langle \bar{\phi} - \phi \rangle = 0$ (see (2.17) and (2.12)), then the effective action satisfies

$$\xi \frac{\partial \Gamma}{\partial \xi}[\bar{\phi}; \xi] + \Gamma_{,j}[\bar{\phi}; \xi] K^j[\bar{\phi}, \xi] = 0, \quad (2.30)$$

with

$$K_j[\bar{\phi}; \xi] = \left\langle \frac{1}{2} \tilde{D}_j^a \tilde{\mathcal{G}}^{ab} \tilde{\mathcal{F}}^b \right\rangle. \quad (2.31)$$

Equation (2.30) is the well-known Nielsen identity of the effective action⁴, expressing the fact that the gauge dependence amounts to a nonlocal field redefinition given by K in (2.31). Although in our derivation we assumed that \mathcal{F}^a did not depend on ξ , it can be seen that if this assumption is relaxed, the formula for K_j becomes

$$K_j[\bar{\phi}; \xi] = \left\langle \frac{1}{2} \tilde{D}_j^a \tilde{\mathcal{G}}^{ab} \tilde{\mathcal{F}}^b - \xi \tilde{D}_j^a \tilde{\mathcal{G}}^{ab} \frac{\partial \tilde{\mathcal{F}}^b}{\partial \xi} \right\rangle. \quad (2.32)$$

As stressed in [6], the gauge-fixing function was kept arbitrary throughout the derivation, the only requirement being that the Faddeev-Popov matrix \mathcal{H} of (2.23) has a well-defined inverse \mathcal{G} .

⁴The expression for the Nielsen identities in [6] involves D_j^a , \mathcal{G}^{ab} and \mathcal{F}^b , rather than their counterparts with tildes. This is because we defined the gauge-fixing function within the path integral in (2.16), while Kobes et al's starting point in reference [6] is obtained from (2.16) after the field redefinition $\phi \rightarrow \bar{\phi} - \phi$. Our choice allows to make a more direct contact with the path integral defining the tunneling rate, and its gauge-fixing.

An immediate consequence of the Nielsen identity is that the value of the effective action on the solutions to the equations of motion,

$$\Gamma_{,i}[\bar{\phi}; \xi] = 0, \quad (2.33)$$

is gauge-independent. We may further use (2.30) to understand how the solutions to (2.33) are affected by a change of the gauge parameter [3]. Let's assume that $\varphi(\xi)$ solves (2.33) for a given ξ . Then, taking a functional derivative with respect to $\bar{\phi}_i$ in (2.30) and imposing (2.33) one gets

$$\left(\xi \frac{\partial}{\partial \xi} + K_j \frac{\delta}{\delta \bar{\phi}_j} \right) \Gamma_{,i}[\bar{\phi}; \xi] \Big|_{\bar{\phi}=\varphi(\xi)} = 0. \quad (2.34)$$

On the other hand, if $\varphi(\xi)$ solves (2.33) for *all* ξ , one should have

$$\xi \frac{d}{d\xi} \Gamma_{,i}[\varphi(\xi); \xi] = \left(\xi \frac{\partial}{\partial \xi} + \xi \frac{d\varphi_j(\xi)}{d\xi} \frac{\delta}{\delta \bar{\phi}_j} \right) \Gamma_{,i}[\bar{\phi}; \xi] \Big|_{\bar{\phi}=\varphi(\xi)} = 0. \quad (2.35)$$

Comparing equations (2.34) and (2.35) allows to conclude that the solutions to the quantum equations of motion lie along the characteristic curve

$$\xi \frac{d\varphi_i(\xi)}{d\xi} = K_i[\varphi(\xi); \xi]. \quad (2.36)$$

The previous results imply that the vacuum-to-vacuum amplitude in the absence of sources is also gauge independent, since $Z[0] = \exp iW[0] = \exp i\Gamma[\varphi(\xi)]$, where $\varphi(\xi)$ is an extremum satisfying $J_j = -\Gamma_{,j}[\varphi(\xi)] = 0$.

The previous derivation of the Nielsen identities can be repeated for the false vacuum effective action functional Γ_F , with the result

$$\xi \frac{\partial \Gamma_F}{\partial \xi}[\bar{\phi}; \xi] + \Gamma_{F,j}[\bar{\phi}; \xi] K_F^j[\bar{\phi}, \xi] = 0, \quad K_{Fj}[\bar{\phi}; \xi] = \left\langle \frac{1}{2} \tilde{D}_j^a \tilde{\mathcal{G}}^{ab} \tilde{\mathcal{F}}^b - \xi \tilde{D}_j^a \tilde{\mathcal{G}}^{ab} \frac{\partial \tilde{\mathcal{F}}^b}{\partial \xi} \right\rangle_F, \quad (2.37)$$

where in this case the false vacuum average can be written as

$$\langle \mathcal{O} \rangle_F^J = e^{-i\Gamma_F} \int_{\bar{\phi}_\infty}^{\bar{\phi}_\infty} [d\phi] \mu(\phi) \mathcal{O} \exp i \left[\tilde{S}_g[\phi; \xi] - \Gamma_{F,j}(\phi^j - \bar{\phi}^j) \right]. \quad (2.38)$$

Once more, it follows that the false vacuum effective action is gauge-independent at its extrema, and the analogue of equation (2.36) holds for the extremal configurations. Furthermore, from the definition of Γ_F as the Legendre transform of W_F one may write the false vacuum transition amplitude in terms of an extremal value of Γ_F ,

$$Z_F[0] = \exp i\Gamma_F[\varphi_F(\xi)], \quad \text{with} \quad J = \Gamma_{F,j}[\varphi_F(\xi)] = 0. \quad (2.39)$$

This implies that $Z_F[0]$ is gauge-independent. From this it automatically follows that the decay rate is gauge-independent as well, as follows from equation (2.10). The formula for the decay rate can be rewritten as

$$\gamma = \frac{2}{VT} \text{Im} \Gamma_F[\varphi_F(\xi)]. \quad (2.40)$$

Note that the tunneling rate is associated to an imaginary part in the false vacuum effective action, which, in contrast to the true-vacuum effective action, is complex rather than real.

Equations (2.39) and (2.40) do not specify the solution that generates an imaginary part for the effective action. In order to further elaborate on this and provide formulae that make contact with Callan and Coleman's calculation of tunneling rates, we may follow their derivation [39], paying particular attention to boundary conditions. The resulting expressions will be valid whenever equation (2.11) holds. Let's assume that a local non-convex effective action Γ_F has been constructed, whose effective potential shows the appearance of a false vacuum configuration $\phi = q_F$. The latter corresponds to a local minimum of the effective potential $V_{F\text{eff}}$, satisfying

$$\left. \frac{\partial V_{F\text{eff}}(\phi; \xi)}{\partial \phi_i} \right|_{\phi=q_F} = 0. \quad (2.41)$$

Note that, given the gauge dependence of V_{eff} , encoded by the Nielsen identities, q_F is itself gauge dependent. Callan and Coleman write the vacuum-to-vacuum transition amplitude as

$$Z_F[0] = \lim_{T \rightarrow \infty} \langle q_F | e^{-iHT} | q_F \rangle = \int_{q_F}^{q_F} [d\phi] \mu(\phi) \exp i \tilde{S}_g[\phi; \xi], \quad (2.42)$$

where implicitly it is assumed that the wave-function of the false state $|F\rangle$ overlaps maximally with the configuration q_F , as we have assumed earlier, and as follows from comparing (2.42) with (2.9). To simplify the treatment of the gauge-dependent boundary conditions, we may rewrite the fields as

$$\phi = \varphi_F(\xi) + \rho, \quad (2.43)$$

where $\varphi_F(\xi)$ is a fixed configuration satisfying equation (2.33), with boundary conditions

$$\lim_{t \rightarrow \pm\infty} \varphi_F = q_F, \quad (2.44)$$

while the field ρ goes to zero at $t = \pm\infty$.

Then the path integral $W_F[0]$ can be rewritten, adding a zero contributions depending on $\Gamma_{,j}[\varphi; \xi] \rho^j = 0$,⁵

$$Z_F[0] = \int_0^0 [d\rho] \mu(\rho) \exp i \left[\tilde{S}_g[\varphi(\xi) + \rho; \xi] - \Gamma_{,j}[\varphi(\xi); \xi] \rho^j \right]. \quad (2.45)$$

As said before, $\tilde{S}_g[\phi; \xi]$ includes the classical action $S[\phi]$, and thus the argument of the exponential in equation (2.45) involves $S[\varphi(\xi) + \rho]$, exactly as in the case of the path integral that defines the effective action evaluated at $\varphi^k(\xi)$ (see (2.21) and (2.22)). Furthermore, the zero boundary conditions in the integral of (2.45) match those of the definition of Γ_F in (2.21). Thus, we recover the relation between $Z_F[0]$ and Γ_F of equation (2.39), identifying the extremal configuration as one that satisfies (2.44). Note that consistency with equation (2.39) demands the gauge-fixing in the path integral in (2.45) to be the same as the one used to calculate the effective action Γ_F in (2.21) and determine the boundary condition q_F by means of equation (2.41). This also follows from the fact that using the gauge-dependent $\varphi^k(\xi)$ in the path integral (2.45) is implicitly assuming that the gauge-fixing enforces the fields to belong to the same slice in the space of orbits of gauge transformations that was chosen for the effective action.

⁵Recall that $\varphi_F(\xi)$ is chosen to satisfy the quantum equations of motion $\Gamma_{,i}[\varphi_F(\xi); \xi] = 0$.

If the path integral of equation (2.45) were to involve a gauge-fixing function $\hat{\mathcal{F}}$ different than the function \mathcal{F} used in the calculation of Γ_F and its extremal configuration $\varphi(\xi)$, gauge independence would be lost. Denoting quantities evaluated in different gauges with a superscript \mathcal{F} or $\hat{\mathcal{F}}$, in this case we would have that $\varphi^{\mathcal{F}}(\xi)$ would not be an extremum of $\Gamma_F^{\hat{\mathcal{F}}}$. Using the Nielsen identity (2.37), it follows that the false-vacuum transition amplitude $Z^{\hat{\mathcal{F}}\mathcal{F}}[0] \equiv \exp i\Gamma_F^{\hat{\mathcal{F}}}[\varphi^{\mathcal{F}}]$ would satisfy

$$\xi \frac{d}{d\xi} Z^{\hat{\mathcal{F}}\mathcal{F}}[0] = i Z^{\hat{\mathcal{F}}\mathcal{F}}[0] \left(\xi \frac{\partial}{\partial \xi} \Gamma_F^{\hat{\mathcal{F}}} + \xi \Gamma_F^{\hat{\mathcal{F}},j} \frac{d\varphi_j^{\mathcal{F}}(\xi)}{d\xi} \right) = i Z^{\hat{\mathcal{F}}\mathcal{F}}[0] \Gamma_F^{\hat{\mathcal{F}},j} \left(K_j^{\mathcal{F}} - K_j^{\hat{\mathcal{F}}} \right), \quad (2.46)$$

which is nonzero unless $\hat{\mathcal{F}} = \mathcal{F}$.

The reader may have noted that, while the usual perturbative calculations of tunneling rates involve an exponential of the classical bounce action, such contribution cannot be readily identified in equation (2.40). The underlying reason is that the extremum of Γ_F is not unique, and a sum over extremal configurations is needed. The origin of this exponential can be made more transparent by modifying the derivation following equation (2.42), after noticing that the fields satisfying the boundary conditions in the path integral in (2.42) belong to different topological classes, labelled by the number of times they “bounce” from q_F to itself between $t = -\infty$ and $t = \infty$, with an infinite time in between bounces. We might then express $Z_F[0]$ in (2.42) as a sum of path integrals $Z_F^{(k)}$ over the different sectors, with boundary conditions q_F^k for each number k of bounces. The Legendre transform of each $Z_F^{(k)}$, associated with time-dependent sources $J^{(k)}$ which give rise to expectation values of the fields inside the k -th class, will define a functional $\Gamma_F^{(k)}$ of the form

$$\exp i\Gamma_F^{(k)}[\bar{\phi}] = \int_0^0 [d\phi]^k \mu(\phi) \exp i \left[S_g[\bar{\phi}, \phi; \xi] - \Gamma_{F,j}^{(k)}[\bar{\phi}; \xi] \phi^j \right], \quad (2.47)$$

where $[d\phi]^k$ denotes that the fields $\phi + \bar{\phi}$ are restricted to the k -th topological class.⁶ Each $\Gamma_F^{(k)}$ satisfies a Nielsen-identity analogous to (2.37), with the averages defined by path integrals within the k -th class.⁷ Within each sector one may define extremal configurations $\varphi^k(\xi)$ satisfying $\Gamma_i^{(k)}[\varphi^k(\xi)] = 0$, and such that $\Gamma^{(k)}[\varphi^k(\xi)]$ is gauge-independent as a consequence of the corresponding Nielsen identity. In analogy with equation (2.43), when expressing $Z_F[0]$ as a sum of path integrals, we may rewrite the fields inside each sector as $\phi = \varphi^k(\xi) + \rho$, where the extremal configuration φ^k satisfies the boundary conditions (2.44), and with the additional constraint that φ^k bounces k times from the vacuum to itself. Then we may write

$$Z_F[0] = \sum_k \int_0^0 [d\rho]^k \mu(\rho) \exp i \left[\tilde{S}_g[\varphi^k(\xi) + \rho; \xi] - \Gamma_{F,j}^{(k)}[\varphi^k(\xi); \xi] \rho^j \right] = \sum_k \exp i\Gamma_F^{(k)}[\varphi^k(\xi)], \quad (2.48)$$

where the last identity follows from equation (2.47) and the previous argument establishing that the gauge-fixing in $\tilde{S}_g[\varphi^k(\xi) + \rho; \xi]$ has to be the same as in $S_g[\varphi^k(\xi), \phi; \xi]$.

In order to perform the sum in k in (2.48), we can resort to the same arguments that were used in [39] to show that the contributions of the ordinary classical k -bounce solutions exponentiate. Since both the effective action and the boundary conditions at $t = \pm\infty$ imposed on the solutions $\varphi^k(\xi)$ are invariant under a finite shift in the time coordinate, then time translations of these solutions

⁶Meaning that ϕ bounces back and forth from zero k times in infinitely spaced time-intervals.

⁷As the boundary conditions did not play a role in the derivation for the Nielsen identities for Γ , Γ_F , one can follow the same reasoning to get identities for $\Gamma_F^{(k)}$.

also solve the quantum equations of motion. This implies that fluctuations of ρ given by arbitrary time translations of φ^k in (2.48) have identical contributions to the path integral. We can define the functional integration on each sector k as an integration over time translations accompanied by a product of k functional integrations of field excitations with time coordinates restricted to lie in between the timestamps $t_1 < t_2 < \dots t_{k-1}$ of the successive completions of the k bounces of φ^k . From the previous arguments it follows that the integration over time translations simply leads to an overall constant

$$\int_{-T/2}^{t_1} d\tilde{t}_1 \int_{t_1}^{t_2} \tilde{t}_2 \dots \int_{t_{k-1}}^{T/2} \tilde{t}_k = \frac{T^k}{k!}.$$

The remaining functional integrations over the time-constrained field excitations factorize, since the time integration in \tilde{S}_g can be written as a sum of integrals that only depend on the field excitations within each time interval. These factorized contributions correspond to path integrals of fluctuations (excluding time translations) around a single bounce. In principle there is an ambiguity in the size of the time intervals chosen for the factorization. All the bounces approach the constant field configuration q_F at the endpoints of the time intervals, so that the ambiguities will disappear if the fluctuations around these endpoints do not contribute. This is guaranteed if the false vacuum effective potential satisfies $V_{F,\text{eff}}[q_F] = 0$. Indeed, the contributions of the field fluctuations over a time stretch \hat{T} in which $\varphi^k(\xi)(t) = q_F$ have the form of a vacuum transition functional $Z_F^{\hat{T}}[q_F]$ defined over the interval \hat{T} . $Z_F^{\hat{T}}$ will have an associated effective action, which will approach Γ_F for $\hat{T} \rightarrow \infty$, and of which the constant configuration q_F is an extremum. Using the same arguments that led to equation (2.39), and recalling that the effective potential is the zero-momentum piece of the effective action, one would conclude that the contributions of fluctuations around the endpoints in between bounces are given by factors of the form $Z_F^{\hat{T}}[q_F] \sim \exp i\Gamma_F[q_F] = \exp(-iT V_{F,\text{eff}}[q_F]) = 1$. The former discussion implies that the factorized path integrals around the bounces will be identical, independently of possible ambiguities in the lengths of the time intervals, so that

$$Z_F^{(k)}[0] = \frac{T^k}{k!} (\tilde{Z}_F[0])^k \Rightarrow \sum_k Z_F^{(k)}[0] = e^{Z_F^1[0]} = \exp \exp i\Gamma_F^{(1)}[\varphi^1(\xi)], \quad (2.49)$$

where we used the relation between $Z_F^{(k)}$ and $\Gamma_F^{(1)}$ that follows from equation (2.48). Finally, putting together (2.40) and (2.49), we arrive to

$$\gamma = -\frac{2}{VT} \text{Im} i e^{i\Gamma_F^{(1)}[\varphi^1(\xi);\xi]}. \quad (2.50)$$

We insist that the former is valid under the assumption $V_{F,\text{eff}}[q_F] = 0$, which generalizes Callan and Coleman's requirement of a zero classical energy for the false vacuum. We can obtain a more familiar-looking expression, and understand how to recover the original results of references [39, 52], by performing an analytic continuation to Euclidean space. Assuming that the analytic continuation from the real time axis to $e^{-i\delta}t$ with $\delta > 0$ is unobstructed by any singularities, one may rotate the integral in S_g to the imaginary axis, and formulate the path integrals in terms of a gauge-fixed Euclidean action,

$$e^{i\Gamma_F^{(1)}[\varphi^1(\xi)]} \equiv e^{-\Gamma_F^{(1)E}[\varphi^{1,E}(\xi)]} = \int_0^1 [d\phi]^1 \mu(\phi) \exp(-S_g^E[\varphi^{1,E}(\xi), \phi; \xi]). \quad (2.51)$$

$S_g^E[\varphi^{1,E}, \phi; \xi]$ is obtained from $-S_g[\varphi^1, \phi; \xi]$ by substituting $t \rightarrow -i\tau$ inside the integrals, and substituting integration in t by integration in τ . In particular, the Euclidean configuration $\varphi^{1,E}(\tau, \vec{x}; \xi)$ is simply given by the analytic continuation of φ^1 to imaginary time, i.e. $\varphi^{1,E}(\tau, \vec{x}; \xi) = \varphi^1(it, \vec{x}; \xi)$. It thus follows that $\varphi^{1,E}$ satisfies the quantum equations of motion of the Euclidean version of the effective action, $\Gamma^{(1),E}$, which can be obtained from $-\Gamma^{(1)}$ doing the same substitutions that allow to get S^E from $-S$. In terms of the Euclidean effective action and an Euclidean time interval $T^E = iT$, equation (2.50) becomes

$$\gamma = \frac{2}{VT^E} \text{Im} e^{-\Gamma_F^{(1),E}[\varphi^{1,E}(\xi); \xi]}. \quad (2.52)$$

An essentially identical formula was obtained in reference [34], arising from a saddle point evaluation of the path integral around a quantum path in theories without gauge fields. In our formalism, the saddle point evaluation around a quantum path is reinterpreted as a way to enforce the appropriate boundary conditions in the path integral, and thus leads to an exact result. Here we have derived it from first principles and clarified that the effective action involved is a non-convex functional associated with the false vacuum, (rather than the usual effective action, associated with the true vacuum), and accounting for field fluctuations which only bounce once from the false vacuum onto itself. We also established the gauge independence of the result, and clarified on the way the subtleties related with the compatibility between boundary conditions and gauge-fixing.

As should be clear from equations (2.51) and (2.52), the usual semiclassical approximation is recovered when in the calculation of the solution $\varphi^{1,E}$ to the equations of motion of $\Gamma^{(1),E}$, the latter is replaced by its classical approximation S^E . In this case, the functional integral of (2.51) becomes identical to the one considered by Callan and Coleman. This integral was interpreted as a saddle point evaluation of the Euclidean version of the path integral in (2.42), though we may look at it in a new light, noticing that it truly represents the lowest order contribution to the tunneling rate in an \hbar expansion, since $\Gamma^E = S^E + O(\hbar)$. The name of “semiclassical approximation” was well earned. An advantage of our exact results (2.50), (2.52) is that they also clarify how quantum corrections should be incorporated, particularly in situations when the tree-level potential has no minima and the saddle point approximation becomes problematic. The path integral has to be indeed evaluated around a background which solves the quantum equations of motion, which validates the methods of [34–36]. An alternative way to get gauge-invariant results for the tunneling rates is to directly compute the false vacuum effective action, including derivative corrections (for example with the methods of references [53] and [54]), and then solve the quantum equations of motion. The effective action can be computed in a gradient expansion. We may consider for example the case of a real scalar σ , setting all other mean fields to zero. Then the derivative expansion will have the form

$$\Gamma_F[\sigma; \xi] = \int d^4x [Z(\sigma; \xi) \partial_\mu \sigma \partial^\mu \sigma - V_{F_{\text{eff}}}(\sigma; \xi) + O(\partial^4)] \equiv \int d^4x \mathcal{L}_{F_{\text{eff}}}. \quad (2.53)$$

The field redefinition K_F appearing in the Nielsen identities will similarly have a gradient expansion [16, 19],

$$K_F(\sigma; \xi) = C(\sigma; \xi) + D(\sigma; \xi) \partial_\mu \sigma \partial^\mu \sigma - \partial^\mu [\tilde{D}(\sigma; \xi) \partial_\mu \sigma] + O(\partial^4). \quad (2.54)$$

Applying these expansions to the Nielsen identity of equation (2.30) yields the identity (1.1) for the effective potential, while for the field renormalization factors one gets [16, 19]

$$\xi \frac{\partial Z}{\partial \xi} = -C \frac{\partial Z}{\partial \sigma} - 2Z \frac{\partial C}{\partial \sigma} + D \frac{\partial V_{F_{\text{eff}}}}{\partial \sigma} + \tilde{D} \frac{\partial^2 V_{F_{\text{eff}}}}{\partial \sigma^2}. \quad (2.55)$$

The identities (1.1) and (2.55) have been used to argue for the gauge independence of tunneling rates in [16]. There it was assumed that the exponential contribution in the usual formulae for tunneling or nucleation rates involved the effective action of the bounce, rather than its classical action, and gauge independence was shown to follow to lowest nontrivial order with vanishing D and \tilde{D} . This did not clarify the situation at higher orders, aside from the fact that the gauge dependence of the fluctuation determinants in the traditional formulae for the tunneling rate was not addressed. Here we have shown that the false vacuum effective action $\Gamma_F^{(1)}$ evaluated at the bounce configuration gives the full answer for the tunneling rate, with no need of including further fluctuation determinants. Moreover the gauge independence of the tunneling rate in the derivative expansion follows trivially from the fact that equations (1.1) and (2.55) yield the following Nielsen identity for $\mathcal{L}_{F_{\text{eff}}}$,

$$\xi \frac{\partial}{\partial \xi} \mathcal{L}_{F_{\text{eff}}} = \frac{\partial \mathcal{L}_{F_{\text{eff}}}}{\partial \sigma} [C + D(\partial\sigma)^2 - \partial^\mu (\tilde{D} \partial_\mu \sigma)] + O(\partial^4), \quad (2.56)$$

which vanishes at a solution to the equations of motion of the effective action, $\frac{\partial \mathcal{L}_{F_{\text{eff}}}}{\partial \sigma} = 0$.

It should be noted that in the path integral in equation (2.42), prior to the field redefinition in (2.43), the correct choice of gauge-fixing function will seem unconventional, obtained from the one used in the effective action Γ_F in equation (2.21) by setting $\bar{\phi} \rightarrow \varphi(\xi)$ and substituting $\phi \rightarrow \phi - \varphi(\xi)$. The need for such a particular gauge-fixing is possibly the reason that the issue of the gauge dependence of tunneling rates has remained obscure for some time. As stressed before, this election of gauge-fixing is enforced by the relation between Z_F and Γ_F through a Legendre transformation, and it ensures that the gauge slices singled out are consistent with the gauge-dependent boundary conditions obtained from the effective action. This becomes more transparent when writing both path integrals in terms of fields satisfying identical boundary conditions ($\phi \rightarrow 0$ at $t \rightarrow \pm\infty$) as in equations (2.21) and (2.48). Then, the choices of gauge-fixings are directly related, and the exact tunneling rate can be elegantly expressed in terms of the effective action, as in equations (2.40), (2.50) and (2.52).

From the previous considerations it follows that the traditional calculations using a saddle point evaluation around a classical Euclidean solution are missing extra gauge dependence from the quantum corrections to the classical bounce, and even if there is an attempt to incorporate them by working with the effective potential rather than the classical one, not only can there be a problem of overcounting quantum corrections, but also an inconsistency in the choice of gauge-fixing.

Finally, we note that our results have a straightforward generalization to finite temperature thermal tunneling, since in this case the effective action and the vacuum-to-vacuum amplitude still have a path integral formulation (see for example the review [55]). This is similar to the Euclidean formulation at zero temperature, but with the fields having (anti)-periodic boundary conditions in the time direction. All the formal manipulations of the path integrals employed to arrive to our results at zero temperature can be reproduced in the finite temperature case.

Although for simplicity we wrote most of the identities in the discrete-sum-approximations of equations (2.7), (2.11), the result of equation (2.40) linking the decay rate of a false vacuum to its associated effective action is valid beyond this simplification, as it just follows from the definition of the Legendre transformation. As argued before, we expect the same general validity for the Nielsen identities and the results concerning gauge-independence derived from them. The formulae (2.50), (2.52), in turn, are only valid in the limit in which $Z_F[0]$ can be approximated by a single path integral, as in equation (2.11). In the most general situation, instead of depending on a $k = 1$ bounce solution with simple boundary conditions fixed by the false vacuum, the tunneling rate can be expressed as an integration over extrema of $k = 1$ effective-action-like functionals with different boundary conditions,

weighed by the false vacuum wave-function ψ_F appearing in (2.8). More explicitly, using the same reasoning leading to (2.50), one has in this case

$$\gamma = -\frac{2}{VT} \operatorname{Re} \log \int [dq][dq'] \mu(q) \mu(q') \psi_F^J(q') \psi_F^{J^*}(q) \exp \exp i \Gamma_F^{(1)q'q}[\varphi^{1,q'q}]. \quad (2.57)$$

In the previous equation, $\Gamma_F^{(1)q'q}$ denotes a functional defined from a path integral analogous to equation (2.21), but with the q_F in the boundary conditions replaced by q', q , and with the integration restricted to field fluctuations in the $k = 1$ class. The configurations $\varphi^{1,q'q}$ are extrema of the $\Gamma_F^{(1)q'q}$, approaching q', q at negative and positive infinite time, respectively, and bouncing only once in between the boundary values.

3 Conclusions

In this paper we have clarified issues concerning the gauge independence of tunneling and nucleation rates, as well as the question of how to consistently incorporate quantum corrections in their calculation. We have also shed light on the role played by effective action functionals, paying attention to their convexity properties. These aspects are relevant for allowing unambiguous physical answers in the study of questions such as the stability of the Standard Model vacuum, or the properties of phase transitions in the early Universe, which can be important for understanding the mechanisms behind baryogenesis.

For some time it has been generally accepted that somehow the quantum effective potential plays a role in the computation of tunneling probabilities. This idea is problematic for two reasons. First, the effective potential is known to be gauge-dependent. Although this dependence cancels out in physical quantities defined at the extrema of the potential, such as vacuum energies and scalar masses, the gauge-dependence could taint the usual computations of tunneling rates, which are sensitive to the values of the potential in between minima. On the other hand, the idea that the effective potential plays a role in quantum tunneling goes against the known fact that the true effective potential of the theory is known to be concave, having thus no false minima.

An understanding to all-orders of how to extract a gauge-independent physical result for tunneling rates was lacking, despite hints in some perturbative calculations. This is also related to the problem of consistently including quantum corrections, which is best illustrated by scenarios in which it is unavoidable to consider quantum corrections to the potential in order to determine the presence of false vacua. It is not straightforward to include these effects in the calculation of tunneling rates using the usual formalism without incurring in a double counting of quantum corrections.

It turns out that the former problems, the gauge dependence of tunneling rates and the consistent inclusion of quantum fluctuations, have a remarkable simple solution. Starting from the false-vacuum transition amplitude onto itself, $Z_F[0]$, it can be seen that the decay rate is exactly determined by a false-vacuum effective action functional, Γ_F , evaluated at a solution to its quantum equations of motion. Gauge-independence is immediate from the fact that the Nielsen identities imply that the value of Γ_F at its extrema does not depend on the gauge parameters. Γ_F differs from the true effective action of the theory, Γ , and is neither convex nor real, so that the associated effective potential $V_{F\text{eff}}$ can have a false vacuum, and Γ_F an imaginary part, without running into inconsistencies. In fact, Γ_F represents one of the “localized” effective actions proposed by Weinberg and Wu [38], where in this case the restriction of the field fluctuations is enforced by the localization of the field-space wave-function of the false vacuum state. In regards to the true-vacuum effective action, we have shown

that the need to sum over path integrals in convex constructions of Γ is due to a multi-peaked wave function of the groundstate.

In Euclidean space, in the approximation in which the false-vacuum transition amplitude reduces to a single path integral, this means

$$\gamma = \frac{2}{VT^E} \text{Im} e^{-\Gamma_F^{(1)E}[\varphi^{1,E}(\xi);\xi]}, \quad (3.1)$$

where T^E is the Euclidean time interval, and the false vacuum effective action $\Gamma_F^{(1)E}$ is defined as the Legendre transform of the contribution $Z_F^{(1)}[0]$ to $Z_F[0]$ which arises from field fluctuations involving a single infinite-time bounce from the false vacuum onto itself. The configuration $\varphi^{1,E}(\xi)$ appearing in (3.1) is a generalized bounce configuration that solves the equation $\Gamma_{F,i}^{(1)E} = 0$. This solution must approach the false vacuum configuration q_F that minimizes the effective potential at Euclidean times $\tau \rightarrow \pm\infty$, and the superscript 1 in $\varphi^{1,E}(\xi)$ reflects the requirement that the field configuration should only bounce once in between the minimum configurations. The effective potential is assumed to be defined in such a way that it vanishes at the false vacuum.

The false vacuum effective action evaluated at the bounce configuration already includes all quantum corrections, and there is no need to include additional fluctuation determinants. From our results it follows that consistent evaluations of tunneling rates can be performed by computing the false-vacuum effective action, including derivative terms (using for example the techniques of [53,54]), and solving for the quantum bounce. Alternatively, one may use the method of external sources of references [35], [35] to directly obtain the effective action evaluated at the bounce (see also [36]). Since the cancellation of the gauge dependence is automatic, much like in the computation of S-matrix elements, there is in principle no need to perform a field redefinition in the effective action to remove the explicit gauge dependence. Rather, consistent physical results arise after properly accounting for derivative terms in the effective action. In a truncated perturbative expansion, order-by-order gauge independence may require appropriate resummations, as it is known to happen with the energies at the minima of the effective potential [20], which are formally gauge-independent. It remains to be seen whether order-by-order gauge independence can be achieved for tunneling rates.

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